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SOME NEW PROBLEMS IN FILTRATION THEORY

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1. Invariant Γ-Integrals in Filtration Theory. The stationary filtration of an incompressible liquid in a homogeneous isotropic porous medium is described by the following equations [1]:

$$\varphi_{,ii} = 0, \quad v_i = \varphi_{,i} \quad (i = 1, 2, 3), \quad \varphi = -(k/\rho g)p - kx_3, \tag{1.1}$$

where v_i are the components of the filtration velocity; p, pressure of the liquid; k, filtration coefficient; ρg, specific weight of the liquid; φ, velocity potential; x_1, x_2, x_3, rectangular Cartesian coordinates (the x_3 axis is directed opposite to the force of gravity).

Let Σ be an arbitrary closed surface in the porous medium under consideration. If within this surface there are no singular points, lines, or surfaces of the field, the following equations hold [2, 3]:

$$\int_\Sigma (v_i, v_i n_k - 2v_i n_i v_k)\, d\Sigma = 0; \tag{1.2}$$

$$\int_\Sigma [(v_i v_i)_{,l}\, n_k - 2\,(v_i, v_k)_{,l}\, n_i]\, d\Sigma = 0 \;\; (i, k, l = 1,2, 3) \ldots, \tag{1.3}$$

where the n_k are the components of the unit normal vector to the surface Σ.

The proof of Eq. (1.2) follows from the transformations

$$\int_\Sigma (v_i v_i n_k - 2v_i v_k n_i)\, d\Sigma = \int_V [(v_i v_i)_{,k} - 2\,(v_i, v_k)_{,i}]\, dV = \int_V (2v_i v_{i,k} - 2v_i v_{k,i} - 2v_{i,i} v_k)\, dV = 0,$$

since, according to (1.1), $v_{i,k} = v_{k,i}$, $v_{i,i} = 0$ over the entire volume V within the surface Σ. The proof of (1.3) and other such equations is analogous.

If within the surface Σ there are singular points, lines, or surfaces of the field, then obviously the left side of Eq. (1.2) will remain unchanged under any deformations of Σ which do not affect the singularities of the field.

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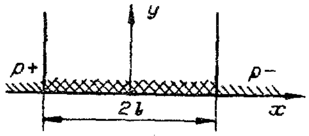

Fig. 1.

We denote by Γ_k the following expressions [2, 3]:

$$\Gamma_k = \frac{\rho}{2\varepsilon^2}\int_{\Sigma}(-v_iv_in_k + 2v_iv_kn_i)\,d\Sigma \quad (k = 1, 2, 3), \tag{1.4}$$

where ρ is the density of the liquid and ε is the porosity of the medium. The quantities Γ_k are invariant characteristics of the singularities of the field that are enclosed within the closed surface Σ. They have the dimensions of force (in the plane case the dimensions of force divided by length). The vector $\Gamma(\Gamma_1, \Gamma_2, \Gamma_3)$ is equal to the principal vector of the configuaration forces acting on an impenetrable body with surface Σ. In the case of a motionless point singularity it yields the force acting from the direction of the field on the indicated singularity. In the case of a point singularity which moves under the influence of the field, its value may be regarded as the dissipation of the field energy expended in moving the singularity by a unit length. The calculation of the Γ-integrals at the singularities of the field is carried out by means of the apparatus of asymptotic Γ-integration [2, 3].

For example, if the singularity is a point source with power q at the origin, so that

$$v_i = qx_i/4\pi r^3 + v_{i0} \quad (r^2 = x_ix_i, \quad i = 1, 2, 3),$$

then in this case [2]

$$\Gamma_k = -\rho\varepsilon^{-2}qv_{k0} \quad (k = 1, 2, 3). \tag{1.5}$$

Here v_{i0} is the regular component of the filtration velocity at the origin.

We give below some obvious Γ-integrals which can be derived from the law of conservation of mass:

$$\int_{\Sigma} v_in_id\Sigma = 0, \int_{\Sigma} v_{i,k}n_id\Sigma = 0 \quad (i, k = 1, 2, 3) \ldots$$

We consider some new problems in the theory of filtration whose effective solution can be obtained by using invariant Γ-integrals. It should be noted that these are powerful calculation devices which enable us to construct easily a finite algebraic system approximately equivalent to the initial boundary-value problem [2] (analogous to the method of finite elements).

2. Theory of Contact Filtration under Dams. Along the concrete foundation of a dam there is often contact filtration, when there is a thin layer of water between the soil and the dam [4]. We shall discuss the theory of this phenomenon by means of a very simple illustrative example. The theory can be generalized without any difficulties of principle to an arbitrary configuration of dam and soil.

Suppose that the soil takes up a half-space $y < 0$, the impermeable dam (the apron) occupies the layer $y > 0$, $|x| < l$, water at pressure p_+ occupies the region $y > 0$, $x < -l$, and water at pressure p_- occupies the region $y > 0$, $x > +l$. For the sake of definiteness, we shall assume that $p_+ > p_-$; this means that the water flows under the dam from left to right (Fig. 1).

The problem is considered to be plane. The fundamental equations of a plane problem, using the complex potential f(z), according to (1.1), can be written in the form

$$v_x + iv_y = \overline{f'(z)}, \quad p = -(\rho g/k)\,\mathrm{Re}\,f(z) - \rho gy \quad (z = x + iy). \tag{2.1}$$

Here the y axis is directed opposite to the force of gravity.

The boundary conditions of this problem have the form

$$\text{for} \quad |x| < l, \; y = 0 \quad v_y = 0, \quad \text{for} \quad |x| > l \quad \partial p/\partial x = 0$$
$$(v_x + iv_y \to 0 \quad \text{as} \quad x + iy \to \infty). \tag{2.2}$$

Hence, on the basis of (2.1), we have the following boundary-value problem for the complex potential:

$$\text{for } \operatorname{Im} z = 0, \quad |\operatorname{Re} z| < l \quad \operatorname{Im} f'(z) = 0,$$
$$\text{for } |\operatorname{Re} z| > l \quad \operatorname{Re} f'(z) = 0 \quad (f'(z) \to 0 \text{ as } z \to \infty), \tag{2.3}$$

and the solution of this problem will be the following:

$$f'(z) = C_i/\sqrt{z^2 - l^2} \quad (\sqrt{z^2 - l^2} \to z \text{ as } z \to \infty). \tag{2.4}$$

The real constant C is determined from the additional condition

$$\int_{-l}^{l} \frac{\partial p}{\partial x}\, dx = \Delta p \, (\Delta p = p_+ - p_-) \text{ for } y = 0. \tag{2.5}$$

Substituting the solution (2.4) into (2.5), in accordance with (2.1), we find

$$C = -k\Delta p/\pi\rho g. \tag{2.6}$$

Near the singular point $z = l$ the solution of (2.4), (2.6) has the form

$$f'(z) = K/\sqrt{2\pi\varepsilon}, \; v_x = -K \sin(\varphi/2)/\sqrt{2\pi|\varepsilon|}, \tag{2.7}$$
$$v_y = K \cos(\varphi/2)/\sqrt{2\pi|\varepsilon|} \quad (\varepsilon = z - l = |\varepsilon| e^{i\varphi}, \quad |\varepsilon| \ll l), \text{ where}$$
$$K = k\Delta p/\rho g\sqrt{\pi l}. \tag{2.8}$$

Using (1.4), (2.7), we calculate the invariant characteristics of the singular point $z = l$:

$$\Gamma_1 = (\rho/2\varepsilon^2)K^2, \quad \Gamma_2 = \Gamma_3 = 0. \tag{2.9}$$

In the present case, according to (2.8), we have

$$\Gamma_1 = k^2(\Delta p)^2/2\rho\varepsilon^2 g^2 \pi l. \tag{2.10}$$

The quantity Γ_1 represents the configuration force of the filtration flow acting on the soil at the singular point $z = l$ and causing all possible critical phenomena in the vicinity of this point (for example, the beginning of contact filtration, the formation of a local cavity in the soil, or, conversely, a bulge in the soil and the breakthrough of a jet of water under the dam). Naturally, the appearance of such critical phenomena, which go beyond the limits of the filtration we are considering, will be characterized by appropriate critical values of the quantity Γ_1. We shall denote by Γ_c the constant characterizing the beginning of contact filtration, so that when $\Gamma_1 < \Gamma_c$, there is no water under the dam, whereas when $\Gamma_1 > \Gamma_c$, a layer of water is formed under the dam and contact filtration takes place.

From this, using (2.10), we find the critical pressure drop across the dam:

$$(\Delta p)_c = (\varepsilon g/k)\sqrt{2\pi\rho\Gamma_c l}. \tag{2.11}$$

The solution constructed above is applicable only for $\Delta p < (\Delta p)_c$. When $\Delta p > (\Delta p)_c$, we must take account of the change in the filtration regime that results from the local erosion of soil particles.

The proposed theory of a breakthrough of water under a dam is applicable only to those problems in which there are singular points. Such points, with trivial exceptions, will always exist — for example, the boundary points of the apron (for arbitrary curvilinear outlines of the apron and channel bottom). According to the microscope principle [2], the filtration field in a neighborhood of these points can always be described by formula (2.7), and the field-intensity coefficient K will be a function of the geometric and physical parameters corresponding to a boundary-value problem of the theory of filtration. An analogous singularity may also be found at the end of a sheet piling, but physically this case is less interesting, since the particles of soil cannot be washed away by the flow of water, and therefore near the end of the piling there will be formed a relatively stable region of nonlinear filtration, in which the structure of the soil particles is different from the initial structure.

We make the following natural physical assumption: the erosion of soil particles at a point of the surface is determined by the filtration velocity at that point. From this assumption it follows that the erosion of the particles always begins at singular points,

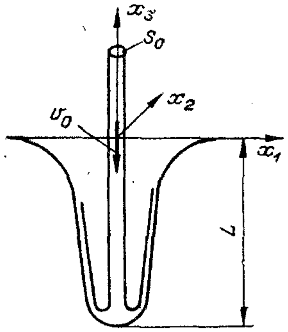

Fig. 2.

where the filtration velocity is infinitely high. Obviously, according to this general assumption, the beginning of particle erosion near a singular point is characterized by some critical value of the field-intensity coefficient at that point, $K = K_c$ (when $K < K_c$, there will be no erosion of the particles). The value of K_c depends on the strength of the bond between the soil particles, on their dimensions and shape, and on the physical properties of the liquid, but it is independent of the macroparameters of the problem (the weight of the dam, the pressure drop, the geometry of the apron, etc.). Therefore for a given combination of soil and liquid, the value of K_c may be determined experimentally (for example, by using a model).

According to the universal relation (2.9), the constants Γ_c and K_c are connected by the following equation:

$$2\varepsilon^2\Gamma_c = \rho K_c^2.$$

Thus, the K_c theory, based on natural physical assumptions, and the Γ_e theory, arising out of the general theory of the motion of the singularities of a physical field [2], lead to identical results.

Problems involving the dynamic interaction of the filtration flow with the soil skeleton are of great importance (see, for example, [1, 4-6]). It should be emphasized that the process by which an individual particle is broken off from the surface of the soil depends only slightly on the macrostresses in the skeleton [6]; it may take place even at very high compressive macrostresses. Physically it is completely different from the process of macrodestruction of the soil skeleton [6].

The breakthrough of water over a dam obviously begins with the elution of soil particles near the right-hand singular point (where $z = l$, in Fig. 1). The further development of the process will not be considered here.

However, if we assume that the transverse dimension of the cavity formed under the dam close to the point $z = l$ as a result of soil erosion is negligibly small in comparison with its length and that the pressure variation in this cavity is also negligibly small, then the nature of the development of the cavity may be appraised by using formula (2.11), in which the value of l should be regarded as a variable parameter. Obviously, on the basis of this relation the process of development of the cavity, once it has begun, will be unstable, since the irreversible reduction of l is accompanyied by a reduction in Δp. The rate dl/dt obvously depends also on the process of transport of the particles in the cavity.

3. Erosion of the Soil by a Jet of Water. Suppose that an axially symmetric jet of water under a high head hits a half-space of porous soil, forming a thin axially symmetric cavity of length L which develops along its axis (Fig. 2). How does L vary with the velocity v_0 of the initial jet and with its cross section S_0?

We shall answer this question on the following assumption, which defines the mathematical model of the phenomenon: all the water from the jet goes into the soil, and the dimension of the region of intensive nonlinear filtration is small in comparison with L.

Thus, we arrive at a problem concerning the motion of a point filtration source of liquid perpendicular to the boundary of the half-space. We shall assume that all of the half-space is saturated with liquid. The effect of the cavity and the deformed boundary of the half-space on the liquid-flow field will be disregarded, which is justifiable in the region far from the cavity and from its free boundary.

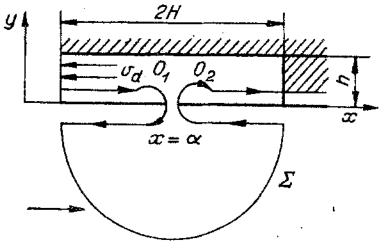

Fig. 3.

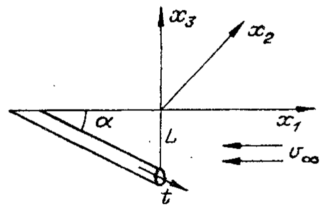

Fig. 4.

The liquid-flow field in the half-space $x_3 < 0$, together with its boundary, under the action of a source with power q at the point (0, 0, −L) is described by the following potential:

$$\varphi = \frac{q}{4\pi}\left[\frac{-1}{\sqrt{x^2 + x_2^2 + (x_3 - L)^2}} + \frac{1}{\sqrt{x_1^2 + x_2^2 + (x_3 + L)^2}}\right]. \tag{3.1}$$

According to (1.5), (3.1), the values of Γ_k at the singular point will have the form

$$\Gamma_1 = \Gamma_2 = 0,\ \Gamma_3 = \rho q^2/(16\pi\varepsilon^2 L^2)\ (q = v_0 S_0). \tag{3.2}$$

The motion of the source takes place under the action of the configuration force Γ_3, which is set equal to the dissipation of the energy expended by the filtration field in displacing the source by a unit length.

We take the simplest concept concerning this quantity and assume that it is constant; we shall denote it by Γ_c. (When $\Gamma_3 < \Gamma_c$, there is no erosion of the soil by the jet of liquid, and the inequality $\Gamma_3 > \Gamma_c$ is impossible according to our concept.) Hence, on the basis of (3.2), we obtain the desired relation expressing the depth of penetration of the jet L as a function of v_0 and S_0:

$$L = (v_0 S_0/4\varepsilon)(\rho/\pi\Gamma_c)^{1/2}.$$

This theory can be easily generalized to other soil shapes and other boundary conditions at the soil surface; the latter factors are essential. In this section we have used only the simplest variant of the general theory of the motion of the singularities [2, 3] of a physical field. A more exact expression in this case will have the form $dL/dt = f(\Gamma_3)$, where t is time and f is an experimentally determined function.

For dense soils, the expression for Γ_3 in (3.2) will contain another term, which represents the contribution of the elastic field of a moving concentrated force.

4. Apron with Drainage Openings. Sometimes drainage openings are made in the aprons in order to reduce the pressure. The debit of an opening depends on its position and structure. Let us consider this question by using the example of Sec. 2, assuming in addition that in the scheme of Fig. 1 there is a sink of liquid of intensity q at the point $y = 0$, $x = a$.

The construction of the drainage opening at this point is shown on a large scale in Fig. 3. It is formed by the impermeable planes $y = h$ and $x = a + H$.

According to (1.2), using the notation of the plane problem, we have*

$$\int_{\Sigma} (-v^2 n_x + 2 v_n v_x)\, d\Sigma = 0, \tag{4.1}$$

where v is the velocity modulus; Σ is the closed contour in Fig. 3. It consists of a semi-neighborhood of radius H much larger than the radius of the opening but much smaller than l, of nonpermeable planes at $y = 0$, $y = h$, and $x = H$, of the plane $x = a - H$ and of two small neighborhoods containing the boundary of the opening. The direction in which the contour Σ is traversed is indicated by the arrow in Fig. 3.

The integral (4.1) vanishes along two small neighborhoods due to the local symmetry of the field, and also due to the fact that these neighborhoods are traversed in the reverse

*In the general case (where the drain is filled with soil with characteristics different from the main region or contains no soil) the relation (4.1) is not exact; this is due to the fact that at the interface between the soil layers the normal components of the filtration velocity vary continuously, while the tangential components are discontinuous.

direction. Integral (4.1) vanishes along the nonpermeable planes at $y = 0$ and $y = h$ since they intersect at $v_n = 0$, $n_x = 0$. On the semineighborhood the terms of the field are described by the potential

$$f'(z) = -q/\pi(z - a) + v_0 + O(z - a), \tag{4.2}$$

where v_0 is a real constant. Making use of (4.2), we can calculate the integral (4.1) along the semicircle; it is found to be equal to $-qv_0$. The integral (4.1) along the impermeable segment $x = a + H_1$ is negligibly small when the value of H_1 is larger than two or three opening diameters, because the velocity decreases exponentially for $x \to \infty$, $y > 0$. The integral (4.1) along the segment $x = a - H_1$, for the same condition, can easily be calculated directly; it is equal to $+qv_d$, where v_d is the filtration velocity when $x \to -\infty$, $y > 0$. It is obvious that $q = v_d h$.

Thus, from Eq. (4.1) we obtain the condition (4.3), $v_d = -v_0$. The potential of the entire field of Fig. 1, with an additional sink at the point $z = a$, will be the following [1]:

$$f'(z) = \frac{-k\Delta p i}{\pi\rho g\sqrt{z^2 - l^2}}\left[1 - \frac{\rho g q\sqrt{l^2 - a^2}}{k\Delta p(z - a)}\right]. \tag{4.3}$$

From this we find the value of v_0 by the formula (4.2):

$$v_0 = k\Delta p/(\pi\rho g\sqrt{l^2 - a^2}). \tag{4.4}$$

Using Eqs. (4.3), (4.4), we find the desired value of the debit of the drainage opening:

$$q = kh\Delta p/(\pi\rho g\sqrt{l^2 - a^2}).$$

The problem concerning this point was solved in [1]. However, that work did not raise the question of determining the flow rate q.

5. Problem of a Drainage Pipe. Suppose that in the half-space $x_3 < 0$ there is a right-cylinder pipe with impermeable walls at an angle α to the boundary of the half-space (Fig. 4). The end of the pipe is at a distance L from the boundary, and it is assumed that $r \ll L$, where r is the radius of the pipe. The unperturbed field of the liquid flow is described by the potential

$$\varphi = -v_\infty x_1. \tag{5.1}$$

The perturbed field far from the end of the pipe is obviously described by the potential corresponding to the point sink q at the end of the pipe, i.e., at the point $(0, 0, -L)$, according to the choice of the coordinate system in Fig. 4:

$$\varphi = \frac{q}{4\pi}\left[\frac{1}{\sqrt{x_1^2 + x_2^2 + (x_3 + L)^2}} \pm \frac{1}{\sqrt{x_1^2 + x_2^2 + (x_3 - L)^2}}\right]. \tag{5.2}$$

Here and hereafter, the upper and lower signs before the second term correspond to the free boundary and the impermeable boundary of the half-space, respectively.

The question arises: what is the value of q? The answer to this question can be obtained by a method analogous to the one used in Sec. 4. According to (1.2), we have

$$\int_\Sigma (v_i v_i n_t - 2v_n v_t)\, d\Sigma = 0,$$

where we take as Σ a closed surface enclosing the end of the pipe and situated at a distance from it which is large in comparison with r but small in comparison with L, plus the inner and outer surfaces of the pipe which are inside the above-mentioned surface.

Calculating the Γ-residue [2, 3], we find

$$v_d = v_{t0}, \tag{5.3}$$

where v_d is the filtration velocity in the tube and v_{t0} is the regular component on the t axis of the filtration velocity corresponding to the external field (5.1), (5.2). We have

$$v_{t0} = v_\infty \cos\alpha \mp q\sin\alpha/16\pi L^2.$$

Since $q = v_d \pi r^2$, we find on the basis of (5.3) that

$$q = \pi r^2 v_\infty \cos\alpha/(1 \mp r^2 \sin\alpha/16L^2).$$

The second term in the denominator of this formula is negligibly small in the approximation considered here.

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